Annales Mathematicae et Informaticae
41 *(2013) pp. 41–55*

Proceedings of the
15th International Conference on Fibonacci Numbers and Their Applications
Institute of Mathematics and Informatics, Eszterházy Károly College
Eger, Hungary, June 25–30, 2012

Identities in the spirit of Ramanujan's amazing identity

Curtis Cooper

Department of Mathematics and Computer Science
University of Central Missouri
Warrensburg, MO 64093
U.S.A.
cooper@ucmo.edu

Abstract

Motivated by an amazing identity by Ramanujan in his "lost notebook", a proof of Ramanujan's identity suggested by Hirschhorn using an algebraic identity, and an algorithm by Chen to find such an algebraic identity, we will establish several identities similar to Ramanujan's amazing identity. For example, if

$$\sum_{n\geq 0} a_n x^n = \frac{9 + 3609x - 135x^2}{1 - 6888x + 6888x^2 - x^3},$$

$$\sum_{n\geq 0} b_n x^n = \frac{10 - 1478x + 172x^2}{1 - 6888x + 6888x^2 - x^3},$$

$$\sum_{n\geq 0} c_n x^n = \frac{12 + 1146x + 138x^2}{1 - 6888x + 6888x^2 - x^3},$$

then

$$a_n^3 + b_n^3 = c_n^3 + 1.$$

Keywords: Ramanujan, identity

MSC: 11A55

1. Introduction

In his "lost notebook", Ramanujan [4] stated the following amazing identity. If

$$\sum_{n\geq 0} a_n x^n = \frac{1 + 53x + 9x^2}{1 - 82x - 82x^2 + x^3},$$

$$\sum_{n\geq 0} b_n x^n = \frac{2-26x-12x^2}{1-82x-82x^2+x^3},$$

$$\sum_{n\geq 0} c_n x^n = \frac{2+8x-10x^2}{1-82x-82x^2+x^3},$$

then

$$a_n^3 + b_n^3 = c_n^3 + (-1)^n.$$

Hirschhorn [2] demonstrated that using the algebraic identity from the "lost notebook",

$$(x^2+7xy-9y^2)^3 + (2x^2-4xy+12y^2)^3 = (2x^2+10y^2)^3 + (x^2-9xy-y^2)^3, \quad (1.1)$$

Ramanujan could have proved his identity. Chen [1] gave an algorithm to produce similar algebraic identities and Ramanujan-like identities. Our goal is to use this procedure to find explicit algebraic identities and Ramanujan-like identities.

2. Third power algebraic identity to Ramanujan-like identity

The following algebraic identity was suggested by Chen [1] and the theorem and proof were suggested by Hirschhorn [2].

Theorem 2.1. *Let*

$$\begin{aligned}&(r_1x^2+s_1xy+t_1y^2)^3 + (r_2x^2+s_2xy+t_2y^2)^3 \\ =&(r_3x^2+s_3xy+t_3y^2)^3 + (x^2-s_4xy-t_4y^2)^3,\end{aligned} \quad (2.1)$$

be an algebraic identity in variables x and y and integer constants r_1, r_2, r_3, s_1, s_2, s_3, s_4, t_1, t_2, t_3, and t_4. Then if

$$\sum_{n\geq 0} a_n x^n = \frac{r_1+(s_1s_4+t_1-r_1t_4)x-t_1t_4x^2}{1-(s_4^2+t_4)x-(s_4^2t_4+t_4^2)x^2+t_4^3x^3},$$

$$\sum_{n\geq 0} b_n x^n = \frac{r_2+(s_2s_4+t_2-r_2t_4)x-t_2t_4x^2}{1-(s_4^2+t_4)x-(s_4^2t_4+t_4^2)x^2+t_4^3x^3},$$

$$\sum_{n\geq 0} c_n x^n = \frac{r_3+(s_3s_4+t_3-r_3t_4)x-t_3t_4x^2}{1-(s_4^2+t_4)x-(s_4^2t_4+t_4^2)x^2+t_4^3x^3}$$

then

$$a_n^3 + b_n^3 = c_n^3 + (-t_4)^{3n}.$$

Proof. Let $w_0 = 0$, $w_1 = 1$, and

$$w_{n+2} = s_4 w_{n+1} + t_4 w_n.$$

The generating function for the sequence $\{w_n\}$ is given by

$$w(x) = \sum_{n\geq 0} w_n x^n = \frac{x}{1 - s_4 x - t_4 x^2}.$$

Now, if $x = w_{n+1}$ and $y = w_n$, then

$$\begin{aligned} x^2 - s_4 xy - t_4 y^2 &= w_{n+1}^2 - s_4 w_{n+1} w_n - t_4 w_n^2 \\ &= w_{n+1}^2 - w_n(s_4 w_{n+1} + t_4 w_n) \\ &= w_{n+1}^2 - w_n w_{n+2} = (-t_4)^n. \end{aligned}$$

The last equality can be proved by induction on n.

Now, let

$$\begin{aligned} a_n &= r_1 x^2 + s_1 xy + t_1 y^2 = r_1 w_{n+1}^2 + s_1 w_{n+1} w_n + t_1 w_n^2, \\ b_n &= r_2 x^2 + s_2 xy + t_2 y^2 = r_2 w_{n+1}^2 + s_2 w_{n+1} w_n + t_2 w_n^2, \\ c_n &= r_3 x^2 + s_3 xy + t_3 y^2 = r_3 w_{n+1}^2 + s_3 w_{n+1} w_n + t_3 w_n^2. \end{aligned}$$

We can show that

$$a_n^3 + b_n^3 = c_n^3 + (-t_4)^{3n}.$$

But, using generating function techniques, we can show that

$$\begin{aligned} \sum_{n\geq 0} w_n^2 x^n &= \frac{x - t_4 x^2}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3}, \\ \sum_{n\geq 0} w_{n+1}^2 x^n &= \frac{1 - t_4 x}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3}, \\ \sum_{n\geq 0} w_n w_{n+1} x^n &= \frac{s_4 x}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{n\geq 0} a_n x^n &= \frac{r_1 + (s_1 s_4 + t_1 - r_1 t_4)x - t_1 t_4 x^2}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4 x^3}, \\ \sum_{n\geq 0} b_n x^n &= \frac{r_2 + (s_2 s_4 + t_2 - r_2 t_4)x - t_2 t_4 x^2}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3}, \\ \sum_{n\geq 0} c_n x^n &= \frac{r_3 + (s_3 s_4 + t_3 - r_3 t_4)x - t_3 t_4 x^2}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3}, \end{aligned}$$

and the proof is complete. □

3. Search for third power algebraic identities

We will attempt to find particular integer constants involving all the r's, s's, and t's which satisfy equation (2.1) with the following procedure.

Procedure to search for third power algebraic identities

1. Pick one particular set of integers r_1, r_2, and r_3 such that
$$r_1^3 + r_2^3 = r_3^3 + 1. \tag{3.1}$$
2. Select a collection of sets of integers t_1, t_2, t_3, and t_4 such that
$$t_1^3 + t_2^3 = t_3^3 - t_4^3. \tag{3.2}$$
Also, select a range of integer values for s_1 and s_2 to search.
   a. For each t_1, t_2, t_3, t_4, s_1, and s_2, compute s_3 and s_4 using the equations
$$\begin{aligned} s_3 &= \frac{s_1 t_1^2 + s_2 t_2^2 + r_1^2 s_1 t_4^2 + r_2^2 s_2 t_4^2}{r_3^2 t_4^2 + t_3^2}, \\ s_4 &= r_3^2 s_3 - r_1^2 s_1 - r_2^2 s_2. \end{aligned}$$
   Make sure these constants can be computed and that they are integers.
   b. Check the following conditions.
$$\begin{aligned} &3r_1 t_1^2 + 3s_1^2 t_1 + 3r_2 t_2^2 + 3s_2^2 t_2 = 3r_3 t_3^2 + 3s_3^2 t_3 + 3t_4^2 - 3s_4^2 t_4, \\ &6r_1 s_1 t_1 + s_1^3 + 6r_2 s_2 t_2 + s_2^3 = 6r_3 s_3 t_3 + s_3^3 + 6s_4 t_4 - s_4^3, \\ &3r_1^2 t_1 + 3r_1 s_1^2 + 3r_2^2 t_2 + 3r_2 s_2^2 = 3r_3^2 t_3 + 3r_3 s_3^2 - 3t_4 + 3s_4^2. \end{aligned}$$
   c. If all the above conditions are satisfied (every equation is true), the resulting collection of r's, s's, and t's form an algebraic identity satisfying equation (2.1).

To prove that the procedure above will produce an algebraic identity, cube the trinomials in (2.1) to obtain

$$\begin{aligned} &t_1^3 y^6 + 3s_1 t_1^2 x y^5 + (3r_1 t_1^2 + 3s_1^2 t_1) x^2 y^4 + (6r_1 s_1 t_1 + s_1^3) x^3 y^3 \\ &\qquad + (3r_1^2 t_1 + 3r_1 s_1^2) x^4 y^2 + 3r_1^2 s_1 x^5 y + r_1^3 x^6 \\ &+ t_2^3 y^6 + 3s_2 t_2^2 x y^5 + (3r_2 t_2^2 + 3s_2^2 t_2) x^2 y^4 + (6r_2 s_2 t_2 + s_2^3) x^3 y^3 \\ &\qquad + (3r_2^2 t_2 + 3r_2 s_2^2) x^4 y^2 + 3r_2^2 s_2 x^5 y + r_2^3 x^6 \\ &= t_3^3 y^6 + 3s_3 t_3^2 x y^5 + (3r_3 t_3^2 + 3s_3^2 t_3) x^2 y^4 + (6r_3 s_3 t_3 + s_3^3) x^3 y^3 \\ &\qquad + (3r_3^2 t_3 + 3r_3 s_3^2) x^4 y^2 + 3r_3^2 s_3 x^5 y + r_3^3 x^6 \\ &- t_4^3 y^6 - 3s_4 t_4^2 x y^5 + (3t_4^2 - 3s_4^2 t_4) x^2 y^4 + (6s_4 t_4 - s_4^3) x^3 y^3 \end{aligned} \tag{3.3}$$

$$+ (-3t_4 + 3s_4^2)x^4y^2 - 3s_4x^5y + x^6.$$

Collecting like terms in (3.3), we obtain the following equation.

$$\begin{aligned}
&(t_1^3 + t_2^3)y^6 + (3s_1t_1^2 + 3s_2t_2^2)xy^5 + (3r_1t_1^2 + 3s_1^2t_1 + 3r_2t_2^2 + 3s_2^2t_2)x^2y^4 \qquad (3.4)\\
&\quad + (6r_1s_1t_1 + s_1^3 + 6r_2s_2t_2 + s_2^3)x^3y^3 + (3r_1^2t_1 + 3r_1s_1^2 + 3r_2^2t_2 + 3r_2s_2^2)x^4y^2\\
&\quad + (3r_1^2s_1 + 3r_2^2s_2)x^5y + (r_1^3 + r_2^3)x^6\\
&= (t_3^3 - t_4^3)y^6 + (3s_3t_3^2 - 3s_4t_4^2)xy^5 + (3r_3t_3^2 + 3s_3^2t_3 + 3t_4^2 - 3s_4^2t_4)x^2y^4\\
&\quad + (6r_3s_3t_3 + s_3^3 + 6s_4t_4 - s_4^3)x^3y^3 + (3r_3^2t_3 + 3r_3s_3^2 - 3t_4 + 3s_4^2)x^4y^2\\
&\quad + (3r_3^2s_3 - 3s_4)x^5y + (r_3^3 + 1)x^6.
\end{aligned}$$

Step 1 in the procedure insures that the coefficients of x^6 in the algebraic identity are equal. In addition, we would like r_1, r_2, and r_3 to be positive integers. For Ramanujan's algebraic identity this condition is trivially true since

$$1^3 + 2^3 = 2^3 + 1.$$

Other trivial values of r_1, r_2, and r_3 which satisfy (3.1) are $r_1 = 1$ and $r_2 = r_3 = r$, where r is a positive integer.

Appendix I gives positive integer values of r_1, r_2, and r_3 ($r_1 < r_2$ and $r_2 \neq r_3$) which satisfy (3.1). These values were determined by a C++ program.

In step 2, we select a collection of t's satisfying (3.4) to try. This guarantees that the coefficients of y^6 in the algebraic identity are equal. In the spirit of Ramanujan, we assume $t_4 = \pm 1$. To obtain nontrivial results, we also require that $t_1 \neq t_2$, $t_1 \neq -t_2$, $t_1 \neq -1$, and $t_2 \neq -1$. Otherwise, some of the t's could be positive or negative integers and cancel each other. Appendix II contains some of the t's which satisfy (3.2). Again, this appendix was constructed with the help of a C++ program.

Also, in step 2 we search a range of integers s_1 and s_2 (via a C++ program). Some typical ranges for s_1 and s_2 were from -1500 to 1500. With the r's, t's, s_1, and s_2 fixed, the constants left are s_3 and s_4. For step 2a, we compute integers s_3 and s_4. The formulas in step 2a are equivalent to the equations equating the coefficients in the xy^5 and x^5y terms in (3.4). These equations are

$$\begin{aligned}
3s_1t_1^2 + 3s_2t_2^2 &= 3s_3t_3^2 - 3s_4t_4^2,\\
3r_1^2s_1 + 3r_2^2s_2 &= 3r_3^2s_3 - 3s_4.
\end{aligned}$$

Step 2a merely solves them for s_3 and s_4 since they are linear equations in those two variables. We also require that $s_4 > 0$.

For step 2b, the conditions we check are the equations resulting from equating the coefficients of the terms x^2y^4, x^3y^3 and x^4y^2 on each side of equation (3.4). In step 2c, if all of these conditions are satisfied, the constants determine an algebraic identity.

4. Third power results

We found the following results. The constants in each row of the following table satisfy (2.1). We include the leading coefficient of 1 in the last trinomial. Recall that the form of the last trinomial is $x^2 - s_4xy - t_4y^2$.

r_1, s_1, t_1	r_2, s_2, t_2	r_3, s_3, t_3	1, s_4, t_4
1,556,-65601	2,-364,83802	2,-36,67402	1,756,1
1,61,-791	2,-40,1010	2,-4,812	1,83,-1
1,7,-9	2,-4,12	2,0,10	1,9,1
1,-25,135	2,-32,138	2,-36,172	1,9,1
1,-227,11161	2,-292,11468	2,-328,14258	1,83,-1
9,412,-11161	10,-180,14258	12,112,11468	1,756,1
9,-126,3753	10,236,-3230	12,96,2676	1,430,-1
9,45,-135	10,-20,172	12,12,138	1,83,-1
9,-169,791	10,-180,812	12,-220,1010	1,9,1
9,-1539,65601	10,-1640,67402	12,-2004,83802	1,83,-1
3753,-126,9	4528,200,-8	5262,84,6	1,430,-1
11161,3481,-791	11468,-1300,1010	14258,1292,812	1,6887,-1
11161,412,-9	11468,-112,12	14258,180,10	1,756,1

The bounds on s_1 and s_2 varied depending on the speed of the search. Note that the third row is the algebraic identity discovered by Ramanujan. This gives Ramanujan's amazing identity. The eighth row gives the algebraic identity

$$\begin{aligned}&(9x^2 + 45xy - 135y^2)^3 + (10x^2 - 20xy + 172y^2)^3\\ =&(12x^2 + 12xy + 138y^2)^3 + (x^2 - 83xy + y^2)^3.\end{aligned}$$

This produces the Ramanujan-like identity result found in the abstract. The seventh row gives the algebraic identity

$$\begin{aligned}&(9x^2 - 126xy + 3753y^2)^3 + (10x^2 + 236xy - 3230y^2)^3\\ =&(12x^2 + 96xy + 2676y^2)^3 + (x^2 - 430y + y^2)^3.\end{aligned}$$

This produces the following Ramanujan-like identity. If

$$\sum_{n\ge 0} a_nx^n = \frac{9 - 54172x + 3753x^2}{1 - 184899x + 184899x^2 - x^3},$$

$$\sum_{n\ge 0} b_nx^n = \frac{10 + 98260x - 3230x^2}{1 - 184899x + 184899x^2 - x^3},$$

$$\sum_{n\ge 0} c_nx^n = \frac{12 + 43968x + 2676x^2}{1 - 184899x + 184899x^2 - x^3},$$

then

$$a_n^3 + b_n^3 = c_n^3 + 1.$$

5. Fourth power algebraic identities to Ramanujan-like identities

McLaughlin [3] found ten sequences whose sums of their first through fifth powers are equal. We will not be so ambitious. The following identity was suggested by Chen [1] and the theorem and proof were suggested by Hirschhorn [2].

Theorem 5.1. *Let*

$$\begin{aligned}&(x^2+s_1xy+t_1y^2)^4+(mx^2+s_2xy+t_2y^2)^4+(nx^2+s_3xy+t_3y^2)^4 \\ =&(mx^2+s_4xy+t_4y^2)^4+(nx^2+s_5xy+t_5y^2)^4+(x^2-s_6xy-t_6y^2)^4,\end{aligned} \tag{5.1}$$

be an algebraic identity in variables x and y and integer constants m, n, s_1, s_2, s_3, s_4, s_5, s_6, t_1, t_2, t_3, t_4, t_5, and t_6. Then if

$$\sum_{n\geq 0} a_nx^n = \frac{1+(s_1s_6+t_1-t_6)x-t_1t_6x^2}{1-(s_6^2+t_6)x-(s_6^2t_6+t_6^2)x^2+t_6^3x^3},$$

$$\sum_{n\geq 0} b_nx^n = \frac{m+(s_2s_6+t_2-mt_6)x-t_2t_6x^2}{1-(s_6^2+t_6)x-(s_6^2t_6+t_6^2)x^2+t_6^3x^3},$$

$$\sum_{n\geq 0} c_nx^n = \frac{n+(s_3s_6+t_3-nt_6)x-t_3t_6x^2}{1-(s_6^2+t_6)x-(s_6^2t_6+t_6^2)x^2+t_6^3x^3},$$

$$\sum_{n\geq 0} d_nx^n = \frac{m+(s_4s_6+t_4-mt_6)x-t_4t_6x^2}{1-(s_6^2+t_6)x-(s_6^2t_6+t_6^2)x^2+t_6^3x^3},$$

$$\sum_{n\geq 0} e_nx^n = \frac{n+(s_5s_6+t_5-nt_6)x-t_5t_6x^2}{1-(s_6^2+t_6)x-(s_6^2t_6+t_6^2)x^2+t_6^3x^3}$$

then

$$a_n^4+b_n^4+c_n^4=d_n^4+e_n^4+(-t_6)^{4n}.$$

Proof. The proof of this theorem is similar to the proof of Theorem 2.1. □

6. Search for fourth power algebraic identities

We will attempt to find particular integer constants involving m, n, and all the s's and t's which satisfy equation (5.1) with the following procedure.

Procedure to search for fourth power algebraic identities

1. Pick one particular set of integers m and n.
2. Select a collection of sets of integers t_1, t_2, t_3, t_4, t_5, and $t_6=\pm 1$ such that $t_1^4+t_2^4+t_3^4=t_4^4+t_5^4+1$. Also, select a range of integer values for s_1, s_2, s_3, and s_4 to search.

a. For each t_1, t_2, t_3, t_4, t_5, t_6, s_1, s_2, s_3, and s_4, compute s_5 and s_6 using the equations

$$s_5 = \frac{s_1t_1^3 + s_2t_2^3 + s_3t_3^3 - s_4t_4^3 - s_1t_6^3 - m^3s_2t_6^3 - n^3s_3t_6^3 + m^3s_4t_6^3}{n^3t_6^3 + t_5^3},$$
$$s_6 = -s_1 - m^3s_2 - n^3s_3 + m^3s_4 + n^3s_5.$$

Make sure these constants can be computed and that they are integers.

b. Check the following conditions.

$$\begin{aligned}
&4t_1^3 + 6s_1^2t_1^2 + 4mt_2^3 + 6s_2^2t_2^2 + 4nt_3^3 + 6s_3^2t_3^2 \\
&\quad = 4mt_4^3 + 6s_4^2t_4^2 + 4nt_5^3 + 6s_5^2t_5^2 - 4t_6^3 + 6s_6^2t_6^2, \\
&12s_1t_1^2 + 4s_1^3t_1 + 12ms_2t_2^2 + 4s_2^3t_2 + 12ns_3t_3^2 + 4s_3^3t_3 \\
&\quad = 12ms_4t_4^2 + 4s_4^3t_4 + 12ns_5t_5^2 + 4s_5^3t_5 - 12s_6t_6^2 + 4s_6^3t_6, \\
&6t_1^2 + 12s_1^2t_1 + s_1^4 + 6m^2t_2^2 + 12ms_2^2t_2 + s_2^4 + 6n^2t_3^2 + 12ns_3^2t_3 + s_3^4 \\
&\quad = 6m^2t_4^2 + 12ms_4^2t_4 + s_4^4 + 6n^2t_5^2 + 12ns_5^2t_5 + s_5^4 + 6t_6^2 - 12s_6^2t_6 + s_6^4, \\
&12s_1t_1 + 4s_1^3 + 12m^2s_2t_2 + 4ms_2^3 + 12n^2s_3t_3 + 4ns_3^3 \\
&\quad = 12m^2s_4t_4 + 4ms_4^3 + 12n^2s_5t_5 + 4ns_5^3 + 12s_6t_6 - 4s_6^3, \\
&4t_1 + 6s_1^2 + 4m^3t_2 + 6m^2s_2^2 + 4n^3t_3 + 6n^2s_3^2 \\
&\quad = 4m^3t_4 + 6m^2s_4^2 + 4n^3t_5 + 6n^2s_5^2 - 4t_6 + 6s_6^2.
\end{aligned}$$

c. If all the above conditions are satisfied (every equation is true), the resulting collection of m, n, s's, and t's form an algebraic identity satisfying equation (5.1).

The proof that this procedure yields an algebraic identity is similar to the previous procedure.

We need to make a couple of remarks. First of all, we pick positive integers m and n with $m < n$. Again, in the spirit of Ramanujan, we assume $t_6 = \pm 1$. We first note that once a solution is found, we have many other similar solutions since every one of the t's could be positive or negative. We list out the nontrivial values of the t's ($1 < t_1 < t_2 < t_3$ and $t_1 \leq t_4$) in Appendix III. This appendix was constructed with the help of a C++ program. Some typical ranges for s_1, s_2, s_3, and s_4 were from -20 to 20. Finally, we require that $s_6 > 0$.

7. Fourth power results

We found the following results. The constants in each row of the following table satisfy (5.1). Again, we include the leading coefficient of 1 in the last trinomial. Recall that the form of the last trinomial is $x^2 - s_6xy - t_6y^2$.

$m = 1$ and $n = 2$

1, s_1, t_1	1, s_2, t_2	2, s_3, t_3	1, s_4, t_4	2, s_5, t_5	1, s_6, t_6
1,-4,4	1,-6,9	2,-10,13	1,-7,11	2,-10,12	1,3,-1
1,-3,4	1.-8,9	2,-11,13	1,-9,12	2,-11,11	1,2,1
1,-1,4	1,-2,9	2,-3,13	1,7,-12	2,-3,-11	1,10,-1
1,-4,5	1,-6,6	2,-10,11	1,-7,9	2,-10,10	1,3,-1
1,0,5	1,-2,6	2,-2,11	1,7,-9	2,-2,-10	1,9,1
1,-4,5	1,-5,6	2,-9,11	1,-7,10	2,-9,9	1,2,1
1,-5,6	1,-10,23	2,-15,29	1,-11,26	2,-15,27	1,4,-1
1,-4,6	1,-12,23	2,-16,29	1,-13,27	2,-16,26	1,3,1
1,0,6	1,-4,23	2,-4,29	1,11,-27	2,-4,-26	1,15,-1
1,-6,7	1,-7,14	2,-13,21	1,-9,18	2,-13,19	1,4,-1
1,-4,7	1,-12,14	2,-16,21	1,-13,19	2,-16,18	1,3,1
1,-4,7	1,0,14	2,-4,21	1,9,-19	2,-4,-18	1,13,-1
1,-7,8	1,-6,11	2,-13,19	1,-9,16	2,-13,17	1,4,-1
1,-5,8	1,-10,11	2,-15,19	1,-11,16	2,-15,17	1,4,-1
1,-3,8	1,0,11	2,-3,19	1,9,-16	2,-3,-17	1,12,1
1,-6,8	1,-6,11	2,-12,19	1,-9,17	2,-12,16	1,3,1

$m = 2$ and $n = 3$

1, s_1, t_1	2, s_2, t_2	3, s_3, t_3	2, s_4, t_4	3, s_5, t_5	1, s_6, t_6
1,-1,7	2,-2,14	3,-3,21	2,10,-19	3,-6,-18	1,16,-1
1,-8,8	2,-10,11	3,-18,19	2,-14,16	3,-17,17	1,3,-1
1,0,8	2,-2,11	3,-2,19	2,10,-16	3,-5,-17	1,15,1
1,-7,8	2,-9,11	3,-16,19	2,-13,17	3,-15,16	1,2,1
1,-8,10	2,-12,19	3,-20,29	2,-16,26	3,-19,25	1,3,1
1,-4,10	2,0,19	3,-4,29	2,12,-26	3,-7,-25	1,19,-1
1,-3,11	2,0,16	3,-3,27	2,12,-23	3,-6,-24	1,18,1
1,0,11	2,-4,39	3,-4,50	2,16,-46	3,-9,-45	1,25,-1
1,-8,13	2,-10,13	3,-18,26	2,-14,22	3,-17,23	1,3,-1
1,4,-13	2,0,-13	3,4,-26	2,4,-22	3,3,-23	1,1,1
1,-1,14	2,-3,41	3,-4,55	2,17,-49	3,-9,-50	1,26,1
1,-8,15	2,-12,19	3,-20,34	2,-16,30	3,-19,29	1,3,1
1,-5,16	2,-1,55	3,-6,71	2,19,-65	3,-11,-64	1,30,-1
1,-6,19	2,0,57	3,-6,76	2,20,-68	3,-11,-69	1,31,1
1,-2,21	2,-6,64	3,10,-113	2,10,-112	3,6,-69	1,22,-1
1,14,-116	2,0,-155	3,14,-271	2,12,-236	3,11,-235	1,1,-1

$m = 3$ and $n = 5$

1, s_1, t_1	3, s_2, t_2	5, s_3, t_3	3, s_4, t_4	5, s_5, t_5	1, s_6, t_6
1,-2,21	3,-4,41	5,6,-71	3,6,-69	5,4,-49	1,22,-1

The bounds on s_1, s_2, s_3, and s_4 varied depending on the speed of the search. The first row of the table for $m = 1$ and $n = 2$ gives the algebraic identity

$$(x^2 - 4xy + 4y^2)^4 + (x^2 - 6xy + 9y^2)^4 + (2x^2 - 10xy + 13y^2)^4$$
$$= (x^2 - 7xy + 11y^2)^4 + (2x^2 - 10xy + 12y^2)^4 + (x^2 - 3xy + y^2)^4.$$

This produces the following Ramanujan-like identity. If

$$\sum_{n\ge 0} a_n x^n = \frac{1 - 7x + 4x^2}{1 - 8x + 8x^2 - x^3},$$
$$\sum_{n\ge 0} b_n x^n = \frac{1 - 8x + 9x^2}{1 - 8x + 8x^2 - x^3},$$
$$\sum_{n\ge 0} c_n x^n = \frac{2 - 15x + 13x^2}{1 - 8x + 8x^2 - x^3},$$
$$\sum_{n\ge 0} d_n x^n = \frac{1 - 9x + 11x^2}{1 - 8x + 8x^2 - x^3},$$
$$\sum_{n\ge 0} e_n x^n = \frac{2 - 16x + 12x^2}{1 - 8x + 8x^2 - x^3},$$

then

$$a_n^4 + b_n^4 + c_n^4 = d_n^4 + e_n^4 + 1.$$

The row in the table for $m = 3$ and $n = 5$ gives the algebraic identity

$$(x^2 - 2xy + 21y^2)^4 + (3x^2 - 4xy + 41y^2)^4 + (5x^2 + 6xy - 71y^2)^4$$
$$= (3x^2 + 6xy - 69y^2)^4 + (5x^2 + 4xy - 49y^2)^4 + (x^2 - 22xy + y^2)^4.$$

This produces the following Ramanujan-like identity. If

$$\sum_{n\ge 0} a_n x^n = \frac{1 - 22x + 21x^2}{1 - 483x + 483x^2 - x^3},$$
$$\sum_{n\ge 0} b_n x^n = \frac{3 - 44x + 41x^2}{1 - 483x + 483x^2 - x^3},$$
$$\sum_{n\ge 0} c_n x^n = \frac{5 + 66x + 71x^2}{1 - 483x + 483x^2 - x^3},$$
$$\sum_{n\ge 0} d_n x^n = \frac{3 + 66x + 69x^2}{1 - 483x + 483x^2 - x^3},$$
$$\sum_{n\ge 0} e_n x^n = \frac{5 + 44x + 49x^2}{1 - 483x + 483x^2 - x^3},$$

then

$$a_n^4 + b_n^4 + c_n^4 = d_n^4 + e_n^4 + 1.$$

8. Questions

The previous data suggests several questions.

1. In the third power case, we were unable to find any nontrivial algebraic identities like (2.1) with $r_1 = 1$ and $r_2 = r_3 = r$ where $r \geq 3$. We would like to know if any exist and if so, what are they?

2. We were unable to find any fourth power algebraic identities of the form

$$\begin{aligned}(r_1x^2 + s_1xy + t_1y^2)^4 + (r_2x^2 + s_2xy + t_2y^2)^4 + (r_3x^2 + s_3xy + t_3y^2)^4 \\ = (r_4x^2 + s_4xy + t_4y^2)^4 + (x^2 - s_5xy - t_5y^2)^4,\end{aligned}$$

   where the r's are positive integers and the s's and t's are nontrivial. Do such identities exist?

3. In the fourth power case, we found algebraic identities for every pair we tried where m if a positive integer and $n = m+1$. Is this always true? In addition, is there any other algebraic identity where $n \neq m + 1$ other than the one we found where $m = 3$ and $n = 5$?

Appendix I: $r_1^3 + r_2^3 = r_3^3 + 1$

r_1	r_2	r_3
9	10	12
64	94	103
73	144	150
135	235	249
244	729	738
334	438	495
368	1537	1544
577	2304	2316
1010	1897	1988
1033	1738	1852
1126	5625	5640
1945	11664	11682
3088	21609	21630
3097	3518	4184
3753	4528	5262
3987	9735	9953
4083	8343	8657
4609	36864	36888
5700	38782	38823
5856	9036	9791
6562	59049	59076
7364	83692	83711
9001	90000	90030
10876	31180	31615
11161	11468	14258
11767	41167	41485
11980	131769	131802
13294	19386	21279
15553	186624	186660
16617	35442	36620

r_1	r_2	r_3
19774	257049	257088
20848	152953	153082
24697	345744	345786
26914	44521	47584
27238	33412	38599
27784	35385	40362
27835	72629	73967
30376	455625	455670
35131	76903	79273
36865	589824	589872
38305	51762	57978
39892	151118	152039
44218	751689	751740
49193	50920	63086
50313	80020	86166
59728	182458	184567
65601	67402	83802
99457	222574	229006
107258	278722	283919
135097	439312	443530
158967	312915	326033
190243	219589	259495
191709	579621	586529
198550	713337	718428
243876	547705	563370
294121	325842	391572
336820	583918	619111
372106	444297	518292
434905	780232	822898
590896	734217	844422

Appendix II: $t_1^3 + t_2^3 = t_3^3 - t_4^3$

t_1	t_2	t_3	t_4
-9	6	-8	1
6	-9	-8	1
-9	8	-6	1
8	-9	-6	1
-8	-6	-9	-1
-6	-8	-9	-1
-8	9	6	-1
9	-8	6	-1
-6	9	8	-1
9	-6	8	-1
6	8	9	1
8	6	9	1
-12	9	-10	-1
9	-12	-10	-1
-12	10	-9	-1
10	-12	-9	-1
-10	-9	-12	1
-9	-10	-12	1
-10	12	9	1
12	-10	9	1
-9	12	10	1
12	-9	10	1
9	10	12	-1
10	9	12	-1
-103	64	-94	-1
64	-103	-94	-1
-103	94	-64	-1
94	-103	-64	-1
-94	-64	-103	1
-64	-94	-103	1
-94	103	64	1
103	-94	64	1
-64	103	94	1
103	-64	94	1
64	94	103	-1
94	64	103	-1

t_1	t_2	t_3	t_4
-144	71	-138	1
71	-144	-138	1
-144	138	-71	1
138	-144	-71	1
-138	-71	-144	-1
-71	-138	-144	-1
-138	144	71	-1
144	-138	71	-1
-71	144	138	-1
144	-71	138	-1
71	138	144	1
138	71	144	1
-150	73	-144	-1
73	-150	-144	-1
-150	144	-73	-1
144	-150	-73	-1
-144	-73	-150	1
-73	-144	-150	1
-144	150	73	1
150	-144	73	1
-73	150	144	1
150	-73	144	1
73	144	150	-1
144	73	150	-1
-172	135	-138	1
135	-172	-138	1
-172	138	-135	1
138	-172	-135	1
-138	-135	-172	-1
-135	-138	-172	-1
-138	172	135	-1
172	-138	135	-1
-135	172	138	-1
172	-135	138	-1
135	138	172	1
138	135	172	1

Appendix III: $t_1^4 + t_2^4 + t_3^4 = t_4^4 + t_5^4 + 1$

t_1	t_2	t_3	t_4	t_5
2	31	47	14	49
2	31	47	49	14
2	35	47	19	50
2	35	47	50	19
2	47	173	71	172
2	47	173	172	71
2	148	191	56	206
2	148	191	206	56
3	6	21	16	19
3	6	21	19	16
3	7	8	2	9
3	7	8	9	2
3	7	44	24	43
3	7	44	43	24
3	21	36	2	37
3	21	36	37	2
3	24	111	77	104
3	24	111	104	77
4	9	13	11	12
4	9	13	12	11
4	18	19	6	22
4	18	19	22	6
4	41	103	58	101
4	41	103	101	58
4	49	75	25	78
4	49	75	78	25
4	76	105	54	110
4	76	105	110	54
4	83	100	32	110
4	83	100	110	32
5	6	11	9	10
5	6	11	10	9
6	14	37	22	36
6	14	37	36	22
6	19	31	9	32
6	19	31	32	9
6	23	29	26	27
6	23	29	27	26

t_1	t_2	t_3	t_4	t_5
6	25	29	15	32
6	25	29	32	15
6	29	47	23	48
6	29	47	48	23
6	31	41	24	43
6	31	41	43	24
6	47	71	43	72
6	47	71	72	43
6	138	165	100	178
6	138	165	178	100
7	14	21	18	19
7	14	21	19	18
7	27	157	109	147
7	27	157	147	109
7	57	73	9	79
7	57	73	79	9
7	76	107	83	104
7	76	107	104	83
7	109	148	121	142
7	109	148	142	121
8	11	19	16	17
8	11	19	17	16
8	43	51	47	48
8	43	51	48	47
8	109	132	62	144
8	109	132	144	62
9	25	34	30	31
9	25	34	31	30
9	34	193	152	171
9	34	193	171	152
9	197	200	45	236
9	197	200	236	45
10	14	103	80	92
10	14	103	92	80
10	19	29	25	26
10	19	29	26	25
10	39	41	32	45
10	39	41	45	32

References

[1] Chen, K.-W., Extensions of an amazing identity of Ramanujan, *The Fibonacci Quarterly*, Vol. 50 (2012), 227–230.

[2] Hirschhorn, M. D., An amazing identity of Ramanujan, *Mathematics Magazine*, Vol. 68 (1995), 199–201.

[3] McLaughlin, J., An identity motivated by an amazing identity of Ramanujan, *The Fibonacci Quarterly*, Vol. 48 (2010), 34–38.

[4] Ramanujan, S., The Lost Notebook and Other Unpublished Papers, New Delhi, Narosa, 1988, p. 341.